MATH 161: Midterm 1

Name: Key

Directions:

- Show your thought process (commonly said as ”show your work”) when solving each problem for full credit.
- If you do not know how to solve a problem, try your best and/or explain in English what you would do.
- Good luck!

Problem	Score	Points
1		10
2		10
3		10
4		10
5		10
6		10
		60

Legend

— my work

— or — my thoughts while I work

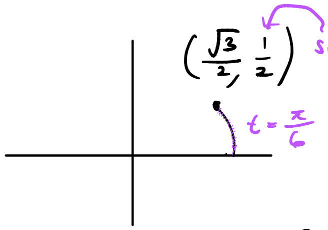
— Common mistakes made to avoid

1. If

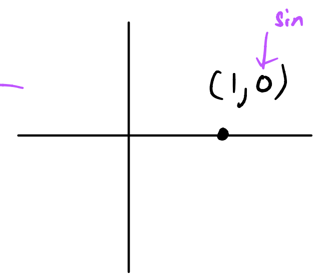
$$f(x) = x^2 - x \qquad g(x) = 3x^2 - x + 1 \qquad h(x) = \sin(x) \qquad j(x) = 2^x$$

Evaluate, expand, and/or simplify the following:

(a) $h\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \boxed{\frac{1}{2}}$



(b) $j(1) \cdot h(0) = 2^1 \cdot \sin(0)$



$= 2 \cdot 0$

$= \boxed{0}$

(c) $\underbrace{f(x)}_{\text{two term}} \cdot \underbrace{g(x)}_{\text{three term}}$

Don't forget parenthesis when multiplying into ≥2 terms!

$$f(x) \cdot g(x) = (x^2 - x)\underbrace{(3x^2 - x + 1)} \overset{\text{dist law}}{=} x^2(3x^2 - x + 1) + (-x)(3x^2 - x + 1)$$

$$\overset{\text{dist law}}{=} 3x^4 \boxed{-x^3} + \boxed{x^2} \boxed{-3x^3} + \boxed{x^2} - x$$

$$= \boxed{3x^4 - 4x^3 + 2x^2 - x}$$

(d) $f(x + h) - f(x)$

Since $f(x) = x^2 - x$

look! x+h $\boxed{\text{replaces}}$ the "x" visually! Now do it!

$$f(\boxed{x+h}) - f(x) = \underbrace{(x+h)^2 - (x+h)}_{f(x+h)} - \underbrace{(x^2 - x)}_{f(x)}$$

Common mistake: forgot the parenthesis!

$$\overset{\text{expand, dist law}}{=} x^2 + 2xh + h^2 - x - h - x^2 + x$$

$$= 2xh + h^2 - h$$

$$\overset{\text{GCF}}{=} \boxed{h(2x + h - 1)}$$

2. Short answer questions:

   (a) Explain in English the intuition (not the definition) behind the symbols $\lim_{x \to a} f(x) = L$.

Skip for midterm 1.

$f(x)$ is the height of the function at an x-value.

So as the x-values approach a but never a itself, the heights $f(x)$ will get closer and closer to the length of L.

   (b) True or false: We can simplify

$$\frac{\underline{3(x-2)^2(x+3)} - 4(x+2)(x-3)^2}{5x(x-3)^2(x-2) - \underline{4(x+3)}}$$

by crossing out the $x+3$.

No. $(x+3)$ is not a factor in the global context of both the numerator and denominator. The context in which it's a factor is underlined in purple above.

   (c) If $f(x) = x - x^2$, evaluate $f(x+h)$ and fully expand + simplify.

this replaces x!

$(A+B)^2$

$$f(\boxed{x+h}) = (x+h) - (x+h)^2$$

Compare the notations:
$f(x+h)$
$f(x)$

look how $x+h$ took the place of x.

$$= x + h - (x^2 + 2xh + h^2) \overset{\text{dist}}{=} \boxed{x + h - x^2 - 2xh - h^2}$$

   (d) If $F(x) = \sin^3(x^2)$ find three functions f, g, h where $f \circ g \circ h = F$.

$$\boxed{\begin{aligned} f(x) &= x^3 \\ g(x) &= \sin(x) \\ h(x) &= x^2 \end{aligned}}$$

Verifying:

deal w/ $h(x)$ first, we know what it is.

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

$$= f(\underline{g(x^2)})$$

x^2 takes the place of x in $g(x)$

$$= f(\overline{\sin(x^2)})$$

$\sin(x^2)$ replaces x in $f(x)$

$$= (\sin(x^2))^3 = \sin^3(x^2) = F(x)$$

Common mistake # 1 : plugging in 2 into $-x^2+1$ is $-2^2+1 = (-1)\cdot 2^2+1$ negative law #1.

$= -4+1$

$= \boxed{-3}$

3. Suppose

$$f(x) = \begin{cases} x & \boxed{x < 1} \\ \boxed{-x^2+1} & x \geq 1 \end{cases}$$

Common mistake # 2 : $x<1$ is

lots of people forgot to take this branch on the interval $(0,1)$. Which means $\boxed{\text{this}}$ part was forgotten.

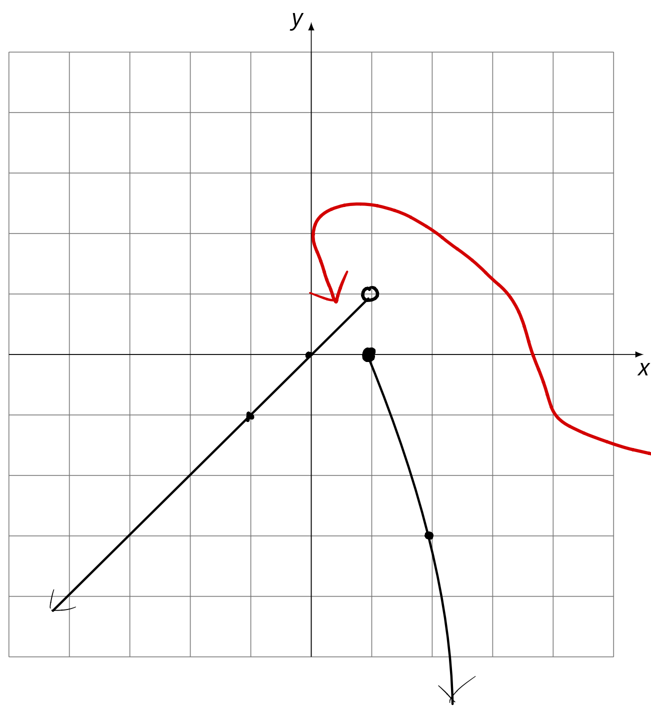
(a) What is $f(1)$?

$f(1) = -1^2+1 \underset{\text{law 1}}{\overset{\text{negative}}{=}} (-1)\cdot 1^2 + 1 = -1+1$

$= \boxed{0}$

(b) Sketch a graph of $f(x)$.

x	$f(x)$
-1	-1
0	0
1	$-1^2+1=0$
2	$-2^2+1=-3$
3	$-3^2+1=-8$



4. Perform the given instruction. Remember to use the relevant laws/properties and **fully simplify**.

   (a) Expand and simplify: $\dfrac{3(x+h)^2 - 1 - (3x^2 - 1)}{h}$

common mistake:

$3(x+h)^2 \neq (3x + 3h)$

because $3(x+h)^2 = 3\cdot(x+h)\cdot(x+h)$

You can only distribute the 3 to one factor of $(x+h)$.

3 multiplies into 3 terms.

Don't forget parenthesis.

$$\frac{3\underbrace{(x+h)^2}_{(A+B)^2} - 1 - (3x^2-1)}{h} \overset{\text{dist}}{=} \frac{3(x^2+2xh+h^2) \cancel{-1} - 3x^2 \cancel{+1}}{h}$$

$$\overset{\text{dist}}{=} \frac{\cancel{3x^2} + 6xh + 3h^2 \cancel{-3x^2}}{h}$$

$$= \frac{6xh + 3h^2}{h} \overset{\text{GCF}}{=} \frac{\cancel{h}(6x+3h)}{\cancel{h}} \underset{\text{law 5}}{\overset{\text{frac}}{=}} \boxed{6x + 3h}$$

   (b) Rationalize the numerator (remember to simplify): $\dfrac{\sqrt{x+h} - \sqrt{x}}{h}$

the only technique to square two terms.

$(A - B)\cdot(A + B) = A^2 - B^2$

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$\underset{\text{law 5}}{\overset{\text{frac}}{=}} \boxed{\frac{1}{\sqrt{x+h} + \sqrt{x}}}$$

mistake #1

$$\left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right)^2 \neq \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h^2}$$

because the numerator are terms and exponents don't interact with terms.

mistake #2

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \sqrt{x+h} + \sqrt{x}$$

forgot denominator $\sqrt{x+h} + \sqrt{x}$

$\frac{2}{x(x+1)}$

Compound fraction.
Technique: get rid of nested denominators by multiplying by LCD of nested denominators.

LCD of $\frac{2}{x(x+1)}, \frac{3}{\sqrt{x}}, \frac{1}{x}$ is $\boxed{x\sqrt{x}(x+1)}$

We begin: (c) Simplify: $\dfrac{\boxed{\dfrac{2}{x^2+x}} - \boxed{\dfrac{3}{\sqrt{x}}}}{\sqrt{x}+\boxed{\dfrac{1}{x}}}$

$$\frac{\overbrace{\frac{2}{x^2+x} - \frac{3}{\sqrt{x}}}^{\text{two terms.}}}{\sqrt{x} + \frac{1}{x}} \cdot \frac{x\sqrt{x}(x+1)}{x\sqrt{x}(x+1)} \underset{\text{law 1.}}{\overset{\text{frac}}{=}} \frac{\left(\frac{2}{x(x+1)} - \frac{3}{\sqrt{x}}\right) x\sqrt{x}(x+1)}{\left(\sqrt{x} + \frac{1}{x}\right) x\sqrt{x}(x+1)}$$

two terms

two terms

$$\underset{\text{law}}{\overset{\text{distributive}}{=}} \frac{\frac{2}{\cancel{x(x+1)}} \cdot \cancel{x(x+1)}\sqrt{x} - \frac{3}{\cancel{\sqrt{x}}} x\cancel{\sqrt{x}}(x+1)}{x(\sqrt{x})^2(x+1) + \frac{1}{\cancel{x}}\cancel{x}\sqrt{x}(x+1)}$$

$$\underset{\text{1, then 5}}{\overset{\text{fraction law}}{=}} \frac{2\sqrt{x} - 3x(x+1)}{x^2(x+1) + \sqrt{x}(x+1)}$$

$x^{\frac{1}{2}} \cdot (x+1) = x^{\frac{1}{2}} \cdot x + x^{\frac{1}{2}}$
$= x^{\frac{3}{2}} + x^{\frac{1}{2}}$
$= \sqrt{x^3} + \sqrt{x}$

$$= \boxed{\frac{2\sqrt{x} - 3x^2 - 3x}{x^3 + x^2 + \sqrt{x^3} + \sqrt{x}}}$$

you could have also factored out $x^{\frac{1}{2}}$ from numerator and denominator, then cancelled.

(d) $\boxed{\text{Expand}}$: $(x^3+6)(2x+1) - (x^2+x-2)(3x^2)$

Convert to terms, no parenthesis.

$$(x^3+6)(2x+1) - (x^2+x-2)(3x^2) \overset{\text{dist}}{=} (x^3+6)2x + (x^3+6)\cdot 1 - 3x^4 - 3x^3 + 6x^2$$

Common mistake: forgetting to distribute the factor of (-1)

$$\overset{\text{dist}}{=} 2x^4 + 12x + x^3 + 6 - 3x^4 - 3x^3 + 6x^2$$

$$= \boxed{-x^4 - 2x^2 + 6x^2 + 12x + 6}$$

5. Determine whether the following sequences is convergent or divergent. If it is convergent, find what the limit converges to.

(a) $a_n = \dfrac{5^n}{5 + 5^n}$

largest "infinite" term in denominator.
Divide both numerator and denominator by 5^n.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5^n}{5+5^n} = \lim_{n\to\infty} \frac{\frac{5^n}{5^n}}{\frac{5+5^n}{5^n}}$$

compound fraction, deal with numerator and denominator separately

$$= \lim_{n\to\infty} \frac{1}{\frac{5}{5^n} + \frac{5^n}{5^n}}$$

frac law 3

$$= \lim_{n\to\infty} \frac{1}{5\cdot\frac{1}{5^n} + 1}$$

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty}\left(5\cdot\frac{1}{5^n}+1\right)}$$

Limit law 5

$$= \frac{1}{5\cdot \lim_{n\to\infty}\frac{1}{5^n} + \lim_{n\to\infty} 1}$$

limit law 6

limit law 3

$$= \frac{1}{5\cdot\lim_{n\to\infty}\left(\frac{1}{5}\right)^n + 1}$$

$$= \frac{1}{5\cdot 0 + 1}$$

$$= \frac{1}{1}$$

$$= \boxed{1}$$

(b) $\boxed{a_n = \dfrac{3^{n+2}}{5^n}}$

try to create r^n so you can use the $\lim_{n\to\infty} r^n$ fact

$$a_n = \frac{3^{n+2}}{5^n} = \frac{3^2\cdot 3^n}{5^n} = 3^2\cdot\frac{3^n}{5^n} = 9\cdot\left(\frac{3}{5}\right)^n$$

LoE 1 and 5

So

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 9\cdot\left(\frac{3}{5}\right)^n$$

$$= 9\cdot\lim_{n\to\infty}\left(\frac{3}{5}\right)^n$$

Limit Law 3

$$= 9\cdot 0$$

$\lim_{n\to\infty} r^n = 0$ if $0<r<1$

$$= \boxed{0}$$

6. Solve the following equations for x:

(a) $e^{2x} - 3e^{x} + 2 = 0$

$(e^{x})^{2} - 3e^{x} + 2 = 0$ Laws of Exponents I

Let $y = e^{x}$. Substituting:

$y^{2} - 3y + 2 = 0$ ← quadratic

$a = 1,\ b = -3,\ c = 2$

new X method: $1\ -2$, $1\ -1$

$(y-2)(y-1) = 0$

$y - 2 = 0$ $\qquad$ $y - 1 = 0$

$y = 2$ $\qquad$ $y = 1$

Now backsubstitute.

$e^{x} = 2$ $\qquad$ $e^{x} = 1$ ← isolated exponential

$\ln(e^{x}) = \ln(2)$ $\qquad$ $\ln(e^{x}) = \ln(1)$

$\boxed{x = \ln(2) \qquad x = \ln(1) = 0}$

(b) $\ln(3x - 10) = 2$

isolated logarithm

$e^{\ln(3x-10)} = e^{2}$

inverse function property

$3x - 10 = e^{2}$

$3x = e^{2} + 10$

$\boxed{x = \frac{e^{2} + 10}{3}}$